DUSO Mathematics League 2015 - 2016

CONTEST #5.

SOLUTIONS

5 - 1. $\boxed{6}$ Solve $3S + 5L = 72$ with $S + L = 20$ to obtain $3(20 - L) + 5L = 72 \to 2L + 60 = 72$, so $L = \mathbf{6}$.

5 - 2. $\boxed{\mathbf{10201}}$ Using laws of exponents, $\left(\sqrt{7} - \sqrt{2}\right)^4 = \left(\left(\sqrt{7} - \sqrt{2}\right)^2\right)^2$, which may be rewritten as $\left(9 - 2\sqrt{14}\right)^2$, which is $9^2 + (2\sqrt{14})^2 - 2 \cdot 9 \cdot 2\sqrt{14} = 137 - 36\sqrt{14}$. Now, $(A - B)^2 = (137 - 36)^2 = (101)^2$, or **10201**.

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5 - 3. $\boxed{\mathbf{1967}}$ The slope of the line is $\dfrac{2015 - 1995}{-2 - 3} = -4$. The point $(10, M)$ also lies on the line, so solve $\dfrac{M - 1995}{10 - 3} = -4$ to obtain $M = \mathbf{1967}$.

5 - 4. $\boxed{\mathbf{22}\pi}$ Notice that $QU^2 + QD^2 = UA^2 + AD^2$, which means that the square of the length of a diameter of circle O is $5^2 + (3\sqrt{7})^2 = \sqrt{43}^2 + (3\sqrt{5})^2 = 88$, so the radius is $\frac{1}{2}\sqrt{88} = \sqrt{22}$. Thus, the area of the circle is $\mathbf{22}\pi$.

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5 - 5. $\boxed{(\mathbf{360}, \mathbf{2})}$ Solve to obtain $(x - 3)\ln 2 = \ln 45 \to x = 3 + \dfrac{\ln 45}{\ln 2}$. Combine the fractions to obtain $x = \dfrac{3\ln 2}{\ln 2} + \dfrac{\ln 45}{\ln 2} = \dfrac{\ln 2^3}{\ln 2} + \dfrac{\ln 45}{\ln 2}$, or $\dfrac{\ln(45 \cdot 8)}{\ln 2}$. The ordered pair is $(\mathbf{360}, \mathbf{2})$.

5 - 6. $\boxed{\dfrac{\sqrt{\mathbf{2}} + \sqrt{\mathbf{6}}}{\mathbf{2}}}$ Note that $BG = BC = 1$. Now, find $m\angle CBG$. The $m\angle GBA = 90°$ and $m\angle CBA = 120°$, so $m\angle CBG = 360 - 90 - 120 = 150°$. Thus, $CG^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 150° = 2 + \sqrt{3}$. But what are P, Q, and R? Squaring $\dfrac{\sqrt{P} + \sqrt{Q}}{R}$ gives $\dfrac{P + Q + 2\sqrt{PQ}}{R^2} = 2 + \sqrt{3}$, so $P + Q = 2R^2$ and $2\sqrt{PQ} = R^2\sqrt{3}$. Because $2\sqrt{PQ} = R^2\sqrt{3}$, and because R is an integer, R^2 is a multiple of 2. So, try $R = 2$. Then, $2\sqrt{PQ} = 4\sqrt{3}$, so $PQ = 12$. Also, $P + Q = 8$, so $P = 2$ and $Q = 6$, and the length $CG = \dfrac{\sqrt{\mathbf{2}} + \sqrt{\mathbf{6}}}{\mathbf{2}}$.

R-1. Compute the least odd positive integer that is the product of four distinct prime numbers.
R-1Sol. $\boxed{\mathbf{1155}}$ Compute $3 \cdot 5 \cdot 7 \cdot 11 = \mathbf{1155}$.

R-2. *Let N be the number you will receive.* The quadratic equation $x^2 - 2x = N$ has two roots. Compute the greater of the two roots.
R-2Sol. $\boxed{\mathbf{35}}$ Substituting, $x^2 - 2x = 1155 \rightarrow (x - 35)(x + 33) = 0$, so the greater root is $x = \mathbf{35}$.

R-3. *Let N be the number you will receive.* In the sequence $41, N, ...$, the difference between any two consecutive terms is constant. Compute the sixth term in the sequence.
R-3Sol. $\boxed{\mathbf{11}}$ The common difference is $N - 41$, so the first six terms are 41, N, $N + (N - 41) = 2N - 41$, $2N - 41 + (N - 41) = 3N - 82$, $3N - 82 + (N - 41) = 4N - 123$, and $4N - 123 + (N - 41) = 5N - 164$. Substituting, the sixth term is $5 \cdot 35 - 164 = \mathbf{11}$.

R-4. *Let N be the number you will receive.* A right circular cylinder has a height of N cm. The surface area of the cylinder (including its top and its bottom) is 120π square cm. Compute the radius of the base of the cylinder in cm.
R-4Sol. $\boxed{\mathbf{4}}$ The surface area of a right circular cylinder is $2\pi rh + 2\pi r^2$, so $2\pi Nr + 2\pi r^2 = 120\pi$, which implies $Nr + r^2 = 60$, which solves to give $r = \dfrac{-N + \sqrt{N^2 + 240}}{2}$. Substituting, $r = \mathbf{4}$.

R-5. *Let N be the number you will receive.* Consider the set $S = \{k, 9, 10, 16, 19, N\}$, where the elements of S are integers. The mean of the numbers in S is 2 more than the median number in S. Compute k.
R-5Sol. $\boxed{\mathbf{32}}$ Substituting, the set $S = \{k, 4, 9, 10, 16, 19\}$. If $k \leq 9$, then the median is 9.5, which makes the mean 11.5, which means $k = 11$, which is not less than or equal to 9. If $9 < k < 10$, a similarly impossible result occurs. If $k = 10$, then the median is 10, so the mean must be 12, which is impossible. If $10 < k < 16$, similarly impossible results occur. So $k \geq 16$, which makes the median 13 and the mean 15, and $k = \mathbf{32}$.

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